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Asymptotic Fixed Point Theorems and Periodic Systems of Functional-Differential Equations

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G. Stephen Jones*

A theorem asserting the existence of a fixed point under a mapping f by utilizing information known about the nth power composite mapping f^n for n sufficiently large, we shall suggestively describe as an asymptotic fixed point theorem. h shall denote the operation defined on the set of subsets of a linear topological space which associates with each such subset its closed convex hull. The following theorem is an example of a useful fixed point theorem of the asymptotic type which is an easy consequence of Tychonoff's theorem [1].

<u>Theorem 1.</u> Let S be a closed convex subset of a complete locally convex linear topological space X and let f be a continuous mapping of S into X with $f(S) \subset S$. If $(fh)^k(S)$ for some positive integer k is contained in a compact set, then f has a fixed point in S.

The difference between the above theorem and the usual Tychonoff theorem is, of course, that the image of S under f is not required to be contained in a compact set. Let us illustrate the usefulness of

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this distinction by an investigation concerning the existence of periodic solutions of differential equations with hereditary dependence. We begin by defining $C(u)$ to be the space of continuous functions mapping a closed interval u into R^n (n-dimensional Euclidean space). A topology on $C(u)$ is specified by the norm functional

$$\|\varphi\| = \sup\{|\varphi_i(\theta)| : i = 1,\ldots, n, \quad \theta \text{ in } u\},$$

defined for each φ in $C(u)$. Let v be a closed interval contained in u and let U be a subset of $C(u)$. U is said to be of compact restriction on v if the functions in U restricted to v form a compact subset of $C(v)$. For each $b > 0$ let $C(u, b) \subset C(u)$ be such that φ in $C(u, b)$ implies $\|\varphi\| \leqq b$. For specified positive constants τ and b_1 let $F(t, \psi)$ be a function mapping $[0, \infty) \times C([-\tau, 0], b_1)$ into R^n which is continuous in each variable separately. We shall relate Theorem 1 to the question of existence of periodic solutions of functional-differential equations of the form

$$\dot{x}(t) = F(t, H_t x), \tag{1}$$

where $H_t x$ denotes the translation to $[-\tau, 0]$ of the restriction of the function x to $[t - \tau, \tau]$. That is, $H_t x(\theta) = x(t + \theta)$ for θ in $[-\tau, 0]$. $\dot{x}(t)$ in equation (1) is understood to denote the right hand derivative of x at t.

Let us assume that corresponding to each initial function in $C_0 = C([-\tau, 0], b_1)$ that equation (1) has a unique solution defined for all $t \geqq 0$. That is, for each φ in C_0 there exists a unique function

x such that the restriction of x to $[-\tau, 0]$ is φ and $\dot{x}(t) = F(t, H_t x)$ for all $t \geq 0$. Furthermore, let us assume that these solutions depend continuously on their corresponding initial functions and that $F(t, \psi)$ is a periodic function of t of period $\omega > 0$ for each fixed ψ in C_o.

Now defining the mapping T for all φ in C_o by the formula

$$T(\varphi)(\theta) = x(\omega + \theta), \quad \theta \text{ in } [-\tau, 0], \tag{2}$$

where x is the solution of equation (1) corresponding to φ, it is clear from our uniqueness assumption for solutions that a fixed point under T implies the existence of a periodic solution for equation (1) of period ω. Let U denote the set of all solutions of equation (1) restricted to $[0, \omega]$ which correspond to initial functions in C_o. If U is contained in a compact subset of $C([0, \omega], b_1)$ and $\omega \geq \tau$, then it is clear from standard theorems that such a fixed point under T exists. However, if $\omega < \tau$ it is only clear that T^k has a fixed point when k is an integer such that $k\omega \geq \tau$. To see that Theorem 1 allows us to drop any restriction on the relationship between τ and ω in concluding the existence of a fixed point under T, we have only to verify that U being contained in a compact subset of $C([0, \omega], b_1)$ implies $(Th)^k(C_o)$ is compact for some integer k. But let k be the smallest integer such that $k\omega \geq \tau$ and observe that for each integer $j \geq 1$ the set $(Th)^j(C_o)$ is of compact restriction on $[-j\omega, 0] \cap [-\tau, 0]$. In particular, therefore, since $k\omega \geq \tau$ we have that $(Th)^k(C_o)$ is contained in a compact subset of C_o. Thus we have proved that the following theorem is a consequence of Theorem 1.

<u>Theorem 2</u>. Let U denote the set of all solutions of equation (1) restricted to $[0, \omega]$ which correspond to initial functions in C_o. If U is contained in a compact subset of $C([0, \omega], b_1)$, then equation (1) has a periodic solution of period ω.

To prove Theorem 1 we simply observe that since S is closed and convex that $fh(S) \subset S$ and consequently $(fh^k)(S)$ is contained in $(fh)^{k-1}(S)$. Hence

$$f(h(fh)^k(S)) \subset (fh)^k(S) \subset h(fh)^k(S).$$

Since it is verified in [2] that the compactness of $(fh)^k(S)$ implies $h(fh)^k(S)$ is compact, the existence of a fixed point under f contained in $(hf)^k(S)$ follows immediately from Tychonoff's theorem.

A theorem which is essentially equivalent to Theorem 2 but which may be more convenient for some purposes is as follows.

<u>Theorem 2'</u>. Let C_1 be a closed convex subset of C_o and let C_2 be a subset of C_1 which is of compact restriction on $[-\omega, 0] \cap [-\tau, 0]$. Suppose that $H_\omega x$ is contained in C_2 for every solution x of equation (1) corresponding to an initial function in C_1. Then equation (1) has a periodic solution of period ω corresponding to an initial function in C_2.

A simple but useful corollary to Theorem 2' may be obtained using a Lipschitz condition to establish compactness.

<u>Corollary 1</u>. Let C_1 be a closed convex subset of C_o and suppose that $H_\omega x$ is contained in C_1 for every solution x of equation (1) corresponding to an initial function in C_1. If there exists a continuous function $L(t)$ such that φ_1 and φ_2 in C_1 and t in $[0, \omega]$ imply

$$|F(t, \varphi_1) - F(t, \varphi_2)| \leq L(t)\|\varphi_1 - \varphi_2\|,$$

then equation (1) has a periodic solution of period ω corresponding to an initial function in C_1.

Proof. Let φ^* be a fixed element in C_1 and let x^* be the corresponding solution of equation (1). For arbitrary φ in C_1 and the corresponding solution x, we have

$$x(t) = \varphi(0) + \int_0^t F(s, H_s x)ds,$$

and

$$x(t) - x^*(t) = \varphi(0) - \varphi^*(0) + \int_0^t (F(s,H_s x) - F(s,H_s x^*))ds.$$

Using our Lipschitz condition for t in $[0, \omega]$ it follows that

$$|x(t) - x^*(t)| \leq |\varphi(0) - \varphi^*(0)| + \int_0^t L(s)\|H_s x - H_s x^*\|ds$$

and

$$\|H_t x - H_t x^*\| \leq \|\varphi - \varphi^*\| + \int_0^t L(s)\|H_s x - H_s x^*\|ds.$$

Hence using Gronwall's lemma we conclude that

$$\|H_t x - H_t x^*\| \leq \|\varphi - \varphi^*\| \exp \{\int_0^t L(s)ds\}.$$

Now for t_1 and t_2 in $[0, \omega]$ we have

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} F(s, H_s x)ds$$

and

$$|x(t_2) - x(t_1)| \leq \int_{t_1}^{t_2} |F(s,H_s x^*)|ds + \int_{t_1}^{t_2} L(s)\|H_s x - H_s x^*\|ds$$

$$\leq \int_{t_1}^{t_2} |F(s,H_s x^*)|ds + \|\varphi - \varphi^*\| \int_{t_1}^{t_2} L(s)\exp\{\int_0^s L(u)du\}ds$$

$$\leq \int_{t_1}^{t_2} |F(s,H_s x^*)|ds + 2b_1[\exp\{\int_0^{t_2} L(s)ds\} - \exp\{\int_0^{t_1} L(s)ds\}].$$

From the integrability of $F(t, H_t x^*)$ and $L(t)$ it follows immediately that the set of solutions of (1) corresponding to initial functions in C_1 and restricted to $[0, \omega]$ form an equicontinuous family of functions. Therefore by the Arzelà-Ascoli theorem this class of restricted solutions is compact, and the application of Theorem 2' completes our proof.

It often happens in the literature that a condition of uniform (with respect to t and the space of initial functions) asymptotic stability for solutions of equations such as (1) is useful in concluding that a transformation such as T (defined by (2)) maps an appropriate set of initial functions into itself. For example, this is the case for the theorems concerned with periodic solutions of perturbed systems contained in [3]. One may observe that in many of these situations the set of initial functions C_o considered under the transformation T may be replaced by a compact set $h(Th)^k(C_o)$ and by so doing an explicit assumption of uniform asymptotic stability is replaceable by an assumption of asymptotic stability only.

Another very interesting asymptotic fixed point theorem which is a significant generalization of the Schauder theorem is presented by Browder in [4] and is as follows:

Browder's Theorem. Let S and S_1 be nonvoid open convex subsets of a Banach space X, S_o a closed convex subset of X, $S_o \subset S_1 \subset S$, f a compact mapping of S into X. Suppose that for a positive integer m, f^m is well-defined on S_1, $\bigcup_{j=0}^{m} f^j(S_o) \subset S_1$, while $f^m(S_1) \subset S_o$. Then f has a fixed point in S_o.

The term compact mapping used in the above theorem implies that f is continuous and f maps S into a compact subset of X. Interesting applications of Browder's theorem may be found in [5] and [6].

Let us now consider the system

$$\dot{x}(t) = X_o[t,x(t), x(t-r)] + \epsilon X_1[t,x(t), x(t-r), \epsilon], \tag{3}$$

where $X_o(t, u, v)$ and $X_1(t, u, v, \epsilon)$ are n-dimensional vector functions continuous in their respective arguments separately, periodic in

t of period ω, and defined for $t \geqq 0$, ϵ small, and $|u| + |v| < M$ for some constant $M > 0$. Furthermore, let us assume that for some positive continuous function $L(t)$ and every (u_1, v_1) and (u_2, v_2) in the domain of X_o,

$$|X_o(t,u_1,v_1) - X_o(t,u_2,v_2)| \leq L(t)(|u_1 - u_2| + |v_1 - v_2|).$$

The following theorem is derived from results presented and proved by using Browder's theorem in [7].

<u>Halanay's Theorem</u>. Let $\omega > r \geqq 0$ and let the equation

$$\dot{y}(t) = X_o[t, y(t), y(t - r)] \tag{4}$$

have a uniformly asymptotically stable periodic solution y_o of period ω such that $|y_o(t)| + |y_o(t - \tau)| < M$ for t in $[0, \omega]$. Suppose that for every continuous function φ defined on $[-r, 0]$ such that $|\varphi(\theta) - y_o(\alpha_o)(\theta)| < b_2$ for θ in $[-r, 0]$ there corresponds a unique solution x of equation (3) and that these solutions depend continuously on their initial functions. Then for ϵ sufficiently small equation (3) has a periodic solution x_ϵ of period ω and $\lim_{\epsilon \to 0} x_\epsilon(t) = y_o(t)$.

Subsequently in this paper we shall present an asymptotic fixed point theorem on a space of continuous functions and construct its proof through the use of Browder's theorem and the Arzelà-Ascoli theorem. We shall then use this new theorem and prove a theorem along the lines of Halanay's theorem but stronger and more general. In particular, we will prove a theorem which is valid for a much broader class of functional-differential equations than represented by (3) and which impose no condition on the relationship between r and ω, and require only the existence of a closed convex discretely asymptotically stably nonincreasing set for the unperturbed equation.

Let I denote the closed interval $[a_1, a_2]$, $a_2 > a_1$, and let Y denote a finite dimensional Banach space. For s in Y let $|s|$ denote the norm of s. $C[I, Y]$ shall denote the set of continuous function mapping I into Y, and $\| \ \|$ denotes the norm functional defined for arbitrary x in $C[I, Y]$ by the formula

$$\|x\| = \max\{|x(t)| : t \text{ in } I\}.$$

<u>Theorem 3</u>. For a specified positive constant b let $X \subset C[I, Y]$ be such that x in X, $t \neq t'$ and t, t' in I imply $|x(t) - x(t')| < b|t - t'|$, and let f be a continuous function mapping an open convex subset B of X into X. Let B_1, B_2, and B_3 be nonvoid bounded open convex subsets of X, B_0 a closed convex subset of $C[I, Y]$ contained in X, and let $B_0 \subset B_1$, $\bar{B}_1 \subset B_2$, and $\bar{B}_2 \subset B_3 \subset B$. If for some positive integer m, f^m is well-defined on B_3, $f^\nu(B_1) \subset B_2$ for $\nu = 1, \ldots, m-1$, and $f^m(B_3) \subset B_0$, then f has a fixed point in B_0.

Let Y^{k+1} denote the set of elements $x = (x_0, \ldots, x_k)$ where x_i is contained in the Banach space Y for $i = 1, 2, \ldots, k$. In preparation to proving Theorem 3 it is convenient to first prove the following lemma which is an easy consequence of Browder's theorem.

Lemma 1. Let $X^* \subset Y^{k+1}$ be the set of all elements $x = (x_0, \ldots, x_k)$ such that $|x_i - x_j| < \frac{\gamma}{k}|i - j|$ for $i \neq j$ and $i,j = 0,1, \ldots, k$ where γ is some positive constant. Let U and U_1 be nonvoid bounded open convex subsets of Y^{k+1} and let V_0 be a closed convex subset such that x in V_0 implies $|x_i - x_j| \leq \frac{\gamma_0}{k}|i - j|$ for $i,j = 0,1, \ldots, k$ where $\gamma > \gamma_0 > 0$. Let $V = U \cap X^*$. $V_1 = U_1 \cap X^*$, and $V_0 \subset V_1 \subset V$. If f is a continuous mapping of V into X^* and for some positive integer m, f^m is well-defined on V_1, $f^\nu(V_0) \subset V_1$ for $\nu = 1, \ldots, m-1$, and $f^m(V_1) \subset V_0$, then f has a fixed point in V_0.

Proof. Let $x = (x_0, \ldots, x_k)$ and $y = (y_0, \ldots, y_k)$ be arbitrary points in X^* and let λ be contained in the interval $[0, 1]$. Then for i,j in $(0,1, \ldots, k)$ we have

$$|[(1-\lambda)x_i + \lambda y_i] - [(1-\lambda)x_j + \lambda y_j]| \leq (1-\lambda)|x_i - x_j| + \lambda|y_i - y_j| < \frac{\gamma}{k}|i - j|.$$

Hence X^* is convex and consequently so are V, V_1 and V_0. Now consider arbitrary x in X^* and let

$$\delta = \max\left\{\frac{k|x_i - x_j|}{|i - j|}\right\}, \quad i \neq j, \quad i,j = 0,1, \ldots, k,$$

and $N(0) = \{z : \|z\| < \frac{\gamma - \delta}{3k}\}$. For y in $x + N(0)$ we have

$$|y_i - y_j| \leq |x_i - x_j| + |z_i - z_j|,$$

where z is in $N(0)$. Hence y in $x + N(0)$ implies

$$|y_i - y_j| < \frac{\gamma}{k}|i - j|$$

for $i \neq j$, $i,j = 0,1, \ldots, k$, so X^* is open and consequently so are V and V_1. Therefore, invoking Browder's theorem it follows immediately that f has a fixed point in V_o and our lemma is proved.

Now proceeding with our proof of Theorem 3 we associate with each positive integer k the partition determined by the set

$$P_k = \{t_i : t_i = a_1 + \tfrac{i}{k}(a_2 - a_1), \quad i = 0,1, \ldots, k\}$$

of the interval $I = [a_1, a_2]$ and for each x in X let $\rho_k(x)$ denote the ordered set

$$(x_o, x_1, \ldots, x_k) = (x(t_o), x(t_1), \ldots, x(t_k)).$$

We define x and y in X as k-equivalent if $\rho_k(x) = \rho_k(y)$ and for each x in X we denote by $g_k(x)$ the set of all elements in X which are k-equivalent to x. Since B_o and $\bar{B}_2$ are closed, uniformly bounded, and equicontinuous families of functions in $C[I, Y]$ we have by a straightforward generalization of the Arzelà-Ascoli theorem as stated in Kolmogorov and Fomin [8] that B_o and $\bar{B}_2$ are compact. Consequently for all k chosen sufficiently large we have

$$S_o = \{g_k(x) : x \text{ in } B_o\} \subset B_1$$

and

$$S_2 = \{g_k(x) : x \text{ in } B_2\} \subset B_3.$$

Hence our hypotheses concerning f imply that for k sufficiently large $f^j(S_o) \subset S_2$ for $j = 1, \ldots, m-1$, and $f^m(S_2) \subset S_o$.

Letting $\Omega_k = \{g_k(x) : x \text{ in } X\}$ we define the mapping $\psi : \Omega_k \to Y^{k+1}$ by the formula

$$\psi(g_k(x)) = \rho_k(x) = x^*.$$

Inducing the topology of Y^{k+1} on Ω_k it is obvious that ψ is a homeomorphism. Since $B \subset X$ and x in X implies $|x(t_1) - x(t_2)| < b\ |t_1 - t_2|$, it is also clear that $B^* = \{\psi(g_k(x)) : x \text{ in } B\}$ is contained in X^* where X^* is the set of all elements x^* in Y^{k+1} such that $|x_i^* - x_j^*| < \frac{\gamma}{k}\ |i - j|$ for $i, j = 0, 1, \ldots, k$, $i \neq j$, and $\gamma = b(a_2 - a_1)$. Let x^* and y^* be arbitrary in B^* and let λ be any number in the interval $[0, 1]$. We have

$$\begin{aligned}(1 - \lambda)x^* + \lambda y^* &= (1 - \lambda)\psi(g_k(x)) + \lambda\ \psi(g_k(y)) \\ &= \psi((1 - \lambda)g_k(x) + \lambda g_k(x)) \\ &= \psi(g_k((1 - \lambda)x + \lambda y)) \\ &= \psi(g_k(z))\end{aligned}$$

where, since B is convex, z is contained in B. Hence $(1 - \lambda)x^* + \lambda y^*$ is contained in B^* and it follows that B^* is convex. Also B open in X clearly implies B^* is open in X^*. Defining $B_0^* = \{\psi(g_k(x)) : x \text{ in } B_0\}$ and $B_2^* = \{\psi(g_k(x)) : x \text{ in } B_2\}$ it follows from essentially the same argument used with B^* that B_0^* is closed, bounded, and convex and B_2^* is open, bounded, and convex.

Now for k sufficiently large we have $f^j\psi^{-1}(B_o^*) = f^j(S_o) \subset S_2$ for $j = 1,2, \ldots, m-1$, and $f^m\psi^{-1}(B_2^*) \subset f^m(S_2) \subset S_o$.

$$\psi f^j\psi^{-1}(B_o^*) = (\psi f\psi^{-1})^j (B_o^*) \subset \psi(S_2) = B_2^*$$

and

$$\psi f^m\psi^{-1}(B_2^*) = (\psi f\psi^{-1})^m (B_2^*) \subset \psi(S_o) = B_o^*.$$

Since B_o is a compact subset of X it follows that there exists a positive number $b_o < b$ such that x in B_o implies $|x(t_i) - x(t_j)| \leq b_o\ |t_i - t_j|$ for $i,j = 0,1, \ldots, k$. Hence it is clear that B_o^* is such that x^* in B_o^* implies $|x_i^* - x_j^*| \leq \frac{\gamma_o}{k}\ |i - j|$ where $\gamma_o = b_o(a_2 - a_1) < \gamma$. We have, therefore, from Lemma 1 that for each k sufficiently large there exist x_k^* in B_o^* such that

$$\psi f\psi^{-1}(x_k^*) = x_k^*$$

and consequently

$$f(\psi^{-1}(x_k^*)) = \psi^{-1}(x_k^*).$$

Thus for each k sufficiently large there exist x_k in B_o such that x_k is contained in $f(g(x_k))$. But considering the sequence $\{x_k\}$ generated as k tends to infinity we see that $\{x_k\}$ being contained in the compact set B_o implies it must contain a subsequence $\{x_p\}$ which converges to a point $\tilde{x}$ in B_o. We also observe that

$$\|\tilde{x} - g_p(x_p)\| \leq \|\tilde{x} - x_p\| + \|x_p - g_p(x_p)\| \leq \|\tilde{x} - x_p\| + \frac{2b}{p}.$$

Hence clearly $g_p(x_p)$ must also converge to $\tilde{x}$. Therefore, since x_p is contained in $f(g(x_p))$ and f is continuous, we may conclude that $f(\tilde{x}) = \tilde{x}$ and our theorem is proved.

We remark that the technique illustrated in Theorem 4 can be very useful in a wide variety of asymptotic problems. For example, the principal result contained in [5] utilizes asymptotic instability and a similar technique. Also a theorem giving a result very much like the result of Theorem 4 but associated with a more general concept of perturbations is presented in [9].

Let V be a subset of C_0 and for every positive number ϵ let $N_\epsilon(V)$ denote the set $\{y : \|y - x\| < \epsilon, \ x \text{ in } V\}$. V will be said to be strictly contained in a set U if there exists a number $\delta > 0$ such that $N_\delta(V) \subset U$. Proceeding now with our discussion of periodic solutions of functional-differential equations we define several notions of stability for sets associated with functional equations of the form of equation (1). First of all a set V is called a discretely stably bounded set of equation (1) if the following condition is satisfied.

(a) There exist a constant $b \geqq 0$ such that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if φ is contained in $N_\delta(V)$ and x is a solution of equation (1) corresponding to φ, then $H_{k\omega}x$ is contained in $N_{b+\epsilon}(V)$ for all positive integers k.

It is clear that if equation (1) has a periodic solution of period ω which is stable in the usual sense, then it has a discretely stably bounded set with $b = 0$ and consisting of a single element. It is also clear that if condition (a) is satisfied then for every positive integer n the mapping T^n, with T defined by (2), sends V into $\overline{N_b(V)}$. Hence if V is closed and convex, $b = 0$, and an appropriate Lipschitz condition is satisfied, the existence of a periodic solution of period ω follows from Corollary 1. If b denotes the smallest nonnegative number for which condition (a) is satisfied, then we will refer to $\overline{N_b(V)}$ as the domain of expansion of V.

A set V is called a discretely asymptotically stably nonincreasing set of equation (1) if condition (a) together with the following condition is satisfied.

(b) There exist a sequence σ of positive integers tending to infinity and a constant $\zeta > 0$ such that if $\overline{N_b(V)}$ is the domain of expansion of V, φ is contained in $N_{b+\zeta}(V)$, x is the solution of equation (1) corresponding to φ, and η is an arbitrary positive constant, then for all integers k in σ and sufficiently large, $H_{k\omega}x$ is contained in $N_\eta(V)$.

We may observe that if equation (1) has a periodic solution of period ω which is asymptotically stable in the usual sense, then it has a discretely asymptotically stably nonincreasing set with $b = 0$, σ the set of all positive integers, and consisting of a single element. We also note that the notion of asymptotically stably nonincreasing sets for functional-differential equations is similar in nature to a condition of ultimate boundedness (in the sense of Yoshizawa) on the solutions of such systems. Sets having the properties of $N_{b+\zeta}(V)$, as specified in (b), will be called domains of discrete asymptotic attraction for V.

A set V is called a uniformly discretely asymptotically stable nonincreasing set of equation (1) if condition (a) together with the following condition is satisfied.

(c) There exist a sequence σ of positive integers tending to infinity and a constant $\zeta > 0$ such that if $\overline{N_b(V)}$ is the domain of expansion of V and η is an arbitrary positive constant, then there exists a positive integer q in σ such that for all integers $k \geqq q$ and contained in σ, $H_{k\omega}x$ is contained in $N_\eta(V)$, where x is a solution of equation (1) corresponding to an initial function in $N_{b+\zeta}(V)$.

As a simple application of Browder's theorem we have the following result.

Theorem 4. Let equation (1) have a convex uniformly discretely asymptotically stably nonincreasing set V with the domain of expansion of its domain of expansion strictly contained in C_0. Furthermore suppose the mapping T as defined by (2) is of compact restriction on $[-\tau, 0] \cap [-\omega, 0]$. Then equation (1) has a periodic solution of period ω.

Proof. Let $\overline{N_b(V)}$ denote the domain of expansion for V and for each $t \geqq 0$ let ξ_t denote the largest integer multiple of ω not exceeding t. Our hypotheses clearly imply the existence of a sequence σ of positive integers tending to infinity and a constant

$\zeta > 0$ such that if φ is contained in $N_{b+\zeta}(V)$ and η is an arbitrary positive constant, then there exists an integer $q \geq \frac{\tau}{\omega}$ and contained in σ such that for all $\xi_t \geq q\omega$ with ξ_t/ω contained in σ and all solutions x of equation (1) corresponding to an initial function in $N_{b+\zeta}(V)$, we have $H_{\xi_t} x$ contained in $N_\eta(V)$. Since $\overline{N_b(V)}$ is strictly contained in C_o, there exists a number δ such that $0 < \delta < \zeta$ and $N_{b+\delta}(V) \subset C_o$. Let δ_1 with $0 < \delta_1 < \delta$ be such that φ in $N_{\delta_1}(V)$ implies $H_{\xi_t} x$ is contained in $N_{b+\delta}(V)$ for all $t \geq 0$. Setting $\eta = \delta_1/2$ and interpreting our observations in terms of the operator T, we may assert that $T^j(N_{\delta_1}(V)) \subset N_{b+\delta}(V)$ for all $j \geq 1$ and $T^q(N_{b+\delta}(V)) \subset N_{\frac{1}{2}\delta_1}(V)$. Furthermore we have by hypothesis that $T^q(N_{b+\delta}(V))$ is compact and by the Mazur theorem so is its closed convex hull. Obviously $h(T^q(N_{b+\delta}(V)))$ is contained in $N_{\delta_1}(V)$, so we have fulfilled the hypotheses for Browder's theorem and can conclude that T has a fixed point φ in $N_{\delta_1}(V)$. It follows, of course, that the solutions of equation (1) having φ as an initial function is periodic of period ω, so the proof of our theorem is complete.

Now defining G to be a function mapping $[0, \infty) \times C_o \times [0, \epsilon_1]$ into R^n, where ϵ_1 is some positive constant, we consider functional-differential equations of the form

$$\dot{x}(t) = G(t, H_t x, \epsilon). \tag{5}$$

$\dot{x}(t)$ and $H_t x$ are defined for equation (1), and we suppose that $G(t, \psi, \epsilon)$ is continuous in each variable separately and periodic in t of period $\omega(\epsilon)$. ω is assumed to be a continuous function on $[0, \epsilon_1]$ and the existence of a positive continuous functional $p(t, \psi, \epsilon)$ defined on $[0, \infty) \times C_o \times [0, \epsilon_1]$ is assumed such that

$$|G(t, \psi, \epsilon) - G(t, \psi, 0)| \leq \epsilon\, p(t, \psi, \epsilon), \tag{6}$$

for ϵ sufficiently small. Furthermore, we assume the existence of a continuous function $L(t)$ such that for arbitrary φ_1 and φ_2 in C_o and t in $[0, \infty)$ we have

$$|G(t, \varphi_1, 0) - G(t, \varphi_2, 0)| \leq L(t)\|\varphi_1 - \varphi_2\|. \tag{7}$$

Theorem 5. Let the equation

$$\dot{z}(t) = G(t, H_t z, 0) \tag{8}$$

have a convex discretely asymptotically stably nonincreasing set V with the domain of expansion of its domain of expansion strictly contained in C_o. Suppose that equation (5) has a unique solution defined for all $t \geq 0$ corresponding to each initial function in C_o and each ϵ in $[0, \epsilon_1]$. Suppose further that these solutions depend continuously on their initial data. Then for each ϵ sufficiently small equation (5) has a periodic solution x_ϵ of period $\omega(\epsilon)$.

Proof. Let $\overline{N_b(V)}$ denote the domain of expansion for V, let $\omega(\epsilon) = \omega_\epsilon$, and for each $t \geqq 0$ let ξ_t denote the largest integer multiple of ω_0 not exceeding t. Our hypotheses clearly imply the existence of a sequence σ of positive integers tending to infinity and a constant $\zeta > 0$ such that if φ is contained in $N_{b+\zeta}(V)$, z is the solution of equation (8) corresponding to φ, and η is an arbitrary positive constant, then for all ξ_t sufficiently large and with ξ_t/ω_0 contained in σ, $H_{\xi_t} z$ is contained in $N_\eta(V)$. Since $\overline{N_b(V)}$ is strictly contained in C_0, there exists a number δ_4 such that $0 < \delta_4 < \zeta$ and $N_{b+\delta}(V) \subset C_0$. Let δ_3, δ_2, and δ_1 be such that $0 < \delta_1 < \delta_2 < \delta_3 < \delta_4$ and φ in $N_{\delta_i}(V)$ implies $H_{\xi_t} z$ is contained in $N_{b+\delta_{i+1}}(V)$ for $i = 1,2,3$ and all $t \geqq 0$. Clearly condition (7) implies that for arbitrary solutions z of equation (8) corresponding to initial functions in $N_{b+\delta_3}(V)$ we have for $t < \xi_t + \omega_0$ that

$$z(t) = z(\xi_t) + \int_{\xi_t}^{t} G(s-\xi_t, H_s z, 0)ds,$$

and

$$\|H_t z\| \leqq \|H_{\xi_t} z\| + \int_{\xi_t}^{t} |G(s - \xi_t, 0, 0)|ds + \int_{\xi_t}^{t} L(s - \xi_t)\|H_s z\|ds.$$

Hence using Gronwall's lemma we have

$$\|H_t z\| \leqq (\|H_{\xi_t} z\| + \int_0^{\omega_0} |G(s, 0, 0)|ds) \exp \{ \int_0^{t - \xi_t} L(s)ds\}.$$

Since $H_{\xi_t} z$ is contained in C_o for all $t \geqq 0$, it follows that $\|H_{\xi_t} z\| \leqq b_1$ and the existence of a constant M_1 such that $\|H_t z\| \leqq M_1$ for all $t \geqq 0$ is clear. Now let t_1 and t_2 be two positive numbers such that $0 < t_2 - t_1 < \omega_o$ but otherwise arbitrary. We have that

$$z(t_2) - z(t_1) = \int_{t_1}^{t_2} G(s - \xi_{t_1}, H_s z, 0) ds,$$

and

$$\begin{aligned} |z(t_2) - z(t_1)| &\leqq \int_{t_1}^{t_2} |G(s - \xi_{t_1}, 0, 0)| ds + \int_{t_1}^{t_2} L(s - \xi_{t_1}) \|H_s z\| ds \\ &\leqq \int_{t_1 - \xi_{t_1}}^{t_2 - \xi_{t_1}} |G(s, 0, 0)| ds + M_1 \int_{t_1 - \xi_{t_1}}^{t_2 - \xi_{t_1}} L(s) ds. \end{aligned}$$

Thus it is also clear that there exist a constant c, independent of the particular choice of t_1 and t_2, such that

$$|z(t_2) - z(t_1)| \leqq c|t_2 - t_1|. \tag{9}$$

Let $K \subset N_{b+\delta_4}(V)$ be the set of all elements φ such that θ_1 and θ_2 in $[-\tau, 0]$ and $\theta_1 \neq \theta_2$ imply $|\varphi(\theta_2) - \varphi(\theta_1)| < 3c|\theta_2 - \theta_1|$. Let

$$K_1 = \{\varphi : \varphi \text{ in } N_{b+\delta_4}(V), |\varphi(\theta_2) - \varphi(\theta_1)| \leqq 2c|\theta_2 - \theta_1|, \theta_1 \text{ and } \theta_2 \text{ in } [-\tau, 0]\}. \tag{10}$$

Since the closure of K is compact we have as a consequence of our asymptotically stably nonincreasing condition that we may choose a constant δ_0 so that $0 < \delta_0 < \delta_1$ and a positive integer q such that φ in K and z the corresponding solution of equation (8) imply that $H_{j\omega_0} z$ is contained in $K_1 \cap N_{\delta_0}(V)$ for all integers $j \geqq q$ and contained in σ.

Now for an arbitrary initial function φ in $\bar{K}$ let us consider the corresponding solution z of equation (8) and the corresponding solution x of equation (5). We have

$$x(t) - z(t) = \int_0^t (G(s, H_s x, \epsilon) - G(s, H_s z, 0))ds,$$

and using (6) and (7) we get

$$|x(t) - z(t)| \leqq \epsilon \int_0^t p(s, H_s x, \epsilon)ds + \int_0^t L(s)\|H_s x - H_s z\|ds.$$

Thus

$$\|H_t x - H_t z\| \leqq \epsilon \int_0^t p(s, H_s x, \epsilon)ds + \int_0^t L(s)\|H_s x - H_s z\|ds,$$

and again employing Gronwall's lemma it follows that

$$\|H_t x - H_t z\| \leqq \epsilon\{\int_0^t p(s,H_s x,\epsilon)ds + \int_0^t L(s)[\int_0^s p(\nu,H_\nu x,\epsilon)d\nu]\exp(\int_s^t L(\nu)d\nu)ds\}. \quad (11)$$

Since our set of initial functions K is conditionally compact it follows that our functions $H_t x$ are contained in a compact set. Thus we may assert the existence of a constant M_2 such that

$$p(t,\ H_t x,\ \epsilon) \leq M_2 \tag{12}$$

for all t in $[0\,,\ q_1\omega_o]$ where q_1 is a fixed integer greater than q and x is allowed to be an arbitrary solution of equation (5) corresponding to an initial function in K. Substituting (12) in (11) we can clearly conclude that there exists a constant M_3 such that for all ϵ sufficiently small and all $t \leq q_1\omega_o$.

$$\|H_t x - H_t z\| \leq \epsilon\ M_3. \tag{13}$$

Thus we can assert the existence of a constant M such that

$$\|H_s x - H_t z\| \leq \|H_s x - H_s z\| + \|H_s z - H_t z\| \leq \epsilon\ M, \tag{14}$$

and

$$\|H_s x\| \leq \epsilon\ M + \|H_t z\|,$$

for all ϵ and $|t - s|$ sufficiently small and $t \leq q_1\omega_o$. In particular we have

$$\|H_{q\omega_\epsilon} x\| \leq \epsilon\ M + \|H_{q\omega_o} z\| \tag{15}$$

for ϵ sufficiently small. In addition, choosing ϵ such that $\epsilon\ M < \frac{\delta_1 - \delta_o}{2}$ it follows that $H_{q\omega_\epsilon} x$ is contained in the neighborhood of $N_{\delta_1}(V)$. Again choosing ϵ sufficiently small and in particular such that $\epsilon\ M < \min\{\frac{\delta_1-\delta_o}{2},\ \frac{\delta_3-\delta_2}{2}\}$, we have by (14) that if x corresponds to an initial function in $N_{\delta_3}(V) \cap K$, then $H_{j\omega_\epsilon} x$ is contained in the neighborhood of $N_{\rho_1}(V)$ for all integers j such that $j\omega_\epsilon \leq q_1\omega_o$ where $\rho_1 = b + \frac{\delta_1+\delta_2}{2}$. Now clearly for t_1 and t_2 in $[0, q_1\omega_o]$ such that $0 < t_2 - t_1 < \omega_o$ we have

$$x(t_2) - x(t_1) = z(t_2) - z(t_1) + \int_{t_1}^{t_2} (G(s, H_s x, \epsilon) - G(s, H_s z, 0))ds,$$

and using (6), (7), (9), (12), and (13) we have

$$\begin{aligned} |x(t_2) - x(t_1)| &\leq c|t_1 - t_2| + \epsilon \int_{t_1}^{t_2} p(s, H_s x, \epsilon)ds + \int_{t_1}^{t_2} L(s)\|H_s x - H_s z\|ds \\ &\leq c|t_1 - t_2| + \epsilon\, M_2\, |t_2 - t_1| + \epsilon\, M_3 L^* |t_2 - t_1|, \end{aligned}$$

where L^* denotes the maximum value of $L(t)$ on $[0, q_1\omega_o]$. Hence obviously ϵ can be chosen sufficiently small so that $H_t x$ for all solutions x of equation (5) corresponding to initial functions in $N_{\delta_3}(V) \cap K$ we have $|x(t_2) - x(t_1)| < 3c\,|t_2 - t_1|$, when t_1 and t_2 are contained in $[-\tau, q_1\omega_o]$ and $t_1 \neq t_2$. That is, $H_o x$ in $N_{\delta_3}(V) \cap K$ implies $H_t x$ is contained in K for all t in $[0, q_1\omega_o]$ when ϵ is chosen sufficiently small. It is clear from the continuity of ω, of course, that $q\omega_\epsilon < q_1\omega_o$ for ϵ chosen sufficiently small. Now let $B_o' = \overline{N_{\frac{\delta_o+\delta_1}{2}}(V)} \cap K_1$, and let B_1', B_2', and B_3' denote the intersections of $N_{\delta_1}(V)$, $N_{\rho_2}(V)$, and $N_{\rho_3}(V)$ respectively with K where $\rho_2 = b + \frac{\delta_1+\delta_2}{2}$ and $\rho_3 = b + \delta_3$.

We now define a mapping T_ϵ on C_o for each ϵ in $[0, \epsilon_1]$ by the formula

$$T_\epsilon(\varphi) = H_{\omega_\epsilon} x,$$

where x, of course, is the solution of equation (5) with $H_0 x = \varphi$. Clearly our hypothesis of continuous dependence on initial data implies T_ϵ is continuous and we have shown that $T_\epsilon^k(B_1') \subset B_2'$ for all positive integers $k \leq q$ and $T_\epsilon^q(B_3') \subset B_0'$ when ϵ is chosen sufficiently small. Furthermore, the sets B_0', B_1', B_2', and B_3' have the same structure as required for the sets B_0, B_1, B_2, and B_3 of Theorem 3 respectively and are interrelated in the same fashion. Therefore it follows from Theorem 3 that T_ϵ has a fixed point φ_ϵ in B_0', and, of course, the solution of equation (5) having φ_ϵ as an initial function must be periodic of period ω_ϵ and our theorem is established.

The question of the "nearness" of periodic solutions of equation (5) and (8) as ϵ becomes small is not in general trivially answered. We do know, however, that the hypotheses of Theorem 5 are not in general sufficient to conclude that an arbitrary periodic solution of equation (8) will have a small neighborhood containing periodic solution of equation (5) for each ϵ chosen sufficiently small. In fact, we can not even conclude the existence of any periodic solutions of equation (8) with this property. Subsequently in this paper we shall indicate in Lemma 2 what can be said with no additional hypotheses and then present several theorems which indicate conditions and types of local behavior in a neighborhood of a specified periodic solution of equation (8) which will imply "nearby" periodic solutions for equation (5) when ϵ is sufficiently small. First, however, let us clarify what we shall mean by two periodic functions with different periods being "near" each other.

We begin by defining for some positive number p_o the set of functions

$$\Omega = \cup \{C([0, p]) : p \text{ in } (0, p_o]\}.$$

For each pair of elements φ and ψ in Ω clearly φ is contained in $C([0, p_1])$ and ψ is contained in $C([0, p_2])$ for some pair of numbers p_1 and p_2 in $[0, p_o]$. $\| \;\|$ is defined for each such pair of elements in Ω by the formula

$$\|\varphi - \psi\| = \sup\{|\varphi(p_1\tau) - \psi(p_2\tau)| : \tau \text{ in } [0, 1]\}. \qquad (17)$$

A periodic function x in $C([0, \infty))$ of period p_1 will be said to be contained in an η-neighborhood, $\eta > 0$, of a periodic function y of period p_2 in $C([0, \infty))$ if the restricted functions $x|p_1$ and $y|p_2$ are such that

$$\|x|p_1 - y|p_2\| < \eta,$$

where $\| \;\|$ is taken in the sense of (17).

<u>Lemma 2</u>. Let the hypotheses of Theorem 5 be satisfied and for each ϵ in $[0, \epsilon_1]$ let Q_ϵ denote the set of periodic solutions of equation (5) of period $\omega(\epsilon)$ corresponding to initial functions in a subset S of C_o contained in a domain of discrete asymptotic attraction of V. For η an arbitrary positive constant, $Q_\epsilon \subset N_\eta (Q_o)$ for

each ϵ sufficiently small.

Proof. It is clear from the proof of Theorem 5 that S may be replaced by K_1 without changing the structure of Q_ϵ when ϵ is chosen sufficiently small. The sets Q_ϵ are obviously compact and defining the mapping

$$\Theta_\epsilon(\varphi) = T_\epsilon(\varphi) - \varphi,$$

we have that $\|\Theta_0(\varphi)\| > 0$ on $K_1 - N_\eta(Q_0)$. Hence since $K_1 - N_\eta(Q_0)$ is compact, it follows that there exist $\mu > 0$ such that $\|\Theta_0(\varphi)\| \geqq \mu$ for x in $K_1 - N_\eta(Q_0)$. If ϵ is chosen sufficiently small we may assert that $\|\Theta_\epsilon(\varphi) - \Theta_0(\varphi)\| < \mu/2$ for all x in $K_1 - N_\eta(Q_0)$, so

$$\|\Theta_\epsilon(\varphi)\| \geqq \|\Theta_0(\varphi)\| - \|\Theta_\epsilon(\varphi) - \Theta_0(\varphi)\| > \mu/2$$

for all φ in $K_1 - N_\eta(Q_0)$. Therefore, it must be that Q_ϵ is contained in $N_\eta(Q_\epsilon)$ for ϵ sufficiently small and our lemma is proved.

Let us now state a theorem which asserts that if x_0 is unique, then perturbed systems of the form of equation (5) have periodic solutions "near" x_0.

<u>Theorem 6</u>. Let the hypotheses of Theorem 5 be satisfied. Furthermore, suppose that x_0 is unique with respect to being a periodic solution of equation (8) of period ω_0 and with an initial function contained in C_0. Then every neighborhood of x_0 contains a periodic solution for equation (5) of period ω_ϵ for each ϵ sufficiently small.

Proof. Let Q_ϵ be as defined in Lemma 2. Then by Lemma 2, for η an arbitrary positive constant, we have that $Q_\epsilon \subset N_\eta(Q_o)$ for each ϵ sufficiently small. But by hypothesis Q_o contains a single element φ_o, so Q_ϵ is contained in $N_\eta(\varphi_o)$. Our theorem now follows at once as a consequence of the continuous dependence of solutions of equation (5) on ϵ and initial data.

One may observe generally that if for η an arbitrary positive constant we have that for each ϵ sufficiently small that $Q_o \subset N_\eta(Q_\epsilon)$, then in every neighborhood of each periodic solution for equation (8) of period ω_o there is at least one periodic solution for equation (5) of period ω_ϵ for each ϵ sufficiently small.

Lemma 3. Let u be a bounded interval, c a positive constant, and let $X \subset C(u)$ be the set of all elements x such that $|x(t_1) - x(t_2)| \leq c|t_1-t_2|$ for all t_1 and t_2 in u. Let L be a continuous linear operator mapping X onto X. Furthermore, let $\delta > 1$ be a constant and let S be a bounded open subset of X which contains the zero element 0. If f is a continuous function mapping S into X for each ϵ in some interval $[0, \epsilon_1]$ and such that

$$f(x, \epsilon) = L(x) + g(x) + \epsilon\, p(x, \epsilon), \tag{18}$$

where $p(x, \epsilon)$ is continuous jointly in its arguments and $g(x) = O(\|x\|^\delta)$ as $\|x\| \to 0$, then for each ϵ sufficiently small $f(S)$ contains a neighborhood of 0 which is open in X.

Proof. L has a continuous linear inverse L^{-1}, so we may write

$$L^{-1}f(x, \epsilon) = x + L^{-1}g(x) + \epsilon\, L^{-1}p(x, \epsilon). \tag{19}$$

Letting f^*, g^*, and p^* denote $L^{-1}f$, $L^{-1}g$, and $L^{-1}p$ respectively we have

$$f^*(x, \epsilon) = x + g^*(x) + \epsilon\, p^*(x, \epsilon).$$

Now let $\mu > 0$ be such that $\overline{N_\mu(0)} \cap X \subset L(S) \cap S$ and let M_1 and M_2 be positive constants such that for all x in $N_\mu(0) \cap X$, $\|g^*(x)\| \leq M_1 \|x\|^\delta$ and $\|p^*(x, \epsilon)\| \leq M_2$. Clearly we may choose $\eta > 0$ so that $\overline{N_\eta(0)} \cap X \subset N_\mu(0) \cap X$ and $M_1\|x\| < \eta/4$ for all x in $\overline{N_\eta(0)} \cap X$. Now selecting an arbitrary element z in $N_{\eta/4}(0) \cap X$ we define the mapping

$$q(x, \epsilon) = z + x - f^*(x, \epsilon) \tag{20}$$

for x in $\overline{N_\eta(0)} \cap X$. Now

$$\begin{aligned}\|q(x, \epsilon)\| &\leq \|z\| + \|g^*(x)\| + \epsilon\|p^*(x, \epsilon)\| \\ &\leq \eta/2 + \epsilon\, M_2,\end{aligned}$$

so clearly for each $\epsilon \leq \frac{\eta}{2M_2}$ we have that q maps $\overline{N_\eta(0)} \cap X$ into itself. Thus by the Schauder fixed point theorem (or for that matter by the Birkhoff-Kellogg fixed point theorem) q has a fixed point in $\overline{N_\eta(0)} \cap X$. That is, for each z in $N_{\eta/4}(0) \cap X$ and each $\epsilon \leq \frac{\eta}{2M_2}$ there exists a point x_1 in $\overline{N_\eta(0)} \cap X$ such that we have

$$x_1 = q(x_1, \epsilon) = z + x_1 - f^*(x_1, \epsilon),$$

which obviously implies $z = f^*(x_1)$. Hence we have shown that $N_{\eta/4}(0) \cap X$ is contained in $f^*(\overline{N_\eta(0)} \cap X)$ for each ϵ sufficiently small. But $N_{\eta/4}(0) \cap X$ contained in $f^*(\overline{N_\eta(0)} \cap X)$ implies

$$L(N_{\eta/4}(0) \cap X) \subset f(\overline{N_\eta(0)} \cap X) \subset f(S),$$

and L being a homeomorphism implies $L(N_{\eta/4}(0) \cap X)$ is open in X. Therefore, the proof of our lemma is complete.

We shall now use Lemma 3 to obtain a rather comprehensive result on the continuity of periodic solutions which should prove useful in applications. The conditions imposed are of such a nature that one might reasonably expect to be able to verify them in physical systems.

Theorem 7. Let the hypotheses of Theorem 5 be satisfied, let X denote the linear extensions of K_1 as defined by (10), and let L be a continuous linear operator mapping X onto X. In addition, let $\delta > 1$ be a constant, let φ_0 be an element in V, and let S be some open subset of X containing φ_0. If for each φ in S

$$H_{\omega_0} z = \varphi + L(\varphi - \varphi_0) + g(\varphi - \varphi_0), \tag{21}$$

where z is the solution of equation (8) corresponding to the initial function φ and $g(\varphi - \varphi_0) = O(\|\varphi - \varphi_0\|^{\delta})$ as $\|\varphi - \varphi_0\| \to 0$, then the solution x_0 of equation (8) corresponding to φ_0 is a periodic solution of period ω_0. Moreover, every neighborhood of x_0 contains a periodic solution for equation (5) of period ω_ϵ for each ϵ sufficiently small.

Proof. It is immediately obvious from (21) that φ_0 is a fixed point under T_0 (as defined by (16)) and consequently x_0 is a periodic solution for equation (8) of period ω_0. We therefore proceed to verify that every neighborhood of x_0 contains a periodic solution for equation (5) of period ω_ϵ when ϵ is sufficiently small. Considering T_ϵ (as defined by (16)) we observe that our hypotheses imply that we may choose a positive constant μ and ϵ sufficiently small so that

$$\begin{aligned} T_\epsilon(\varphi) &= T_0(\varphi) + T_\epsilon(\varphi) - T_0(\varphi) \\ &= \varphi + L(\varphi - \varphi_0) + g(\varphi - \varphi_0) + (T_\epsilon(\varphi) - T_0(\varphi)), \end{aligned} \tag{22}$$

for all φ in $N_\mu(\varphi_0) \cap X$. Furthermore, (14) implies that μ may be selected so that

$$T_\epsilon(\varphi) - T_0(\varphi) = \epsilon\, q(\varphi, \epsilon) \tag{23}$$

for all φ in $N_\mu(\varphi_0) \cap X$ and ϵ sufficiently small where $q(\varphi, \epsilon)$ is continuous jointly in its arguments. Now define

$$\Theta_\epsilon(\psi) = L(\psi) + g(\psi) + \epsilon\, q^*(\psi, \epsilon), \tag{24}$$

for ψ in $N_\mu(0)$ where $q^*(\psi, \epsilon) = q(\varphi_0 + \psi, \epsilon)$, and let η be an arbitrary positive constant. By Lemma 3 we have that there exist a positive constant ξ such that

$$N_\xi(0) \subset \Theta_\epsilon(N_\eta(0)),$$

so in particular 0 is contained in $\Theta_\epsilon(N_\eta(0))$. That is, there is a point ψ_1 in $N_\eta(0)$ such that

$$L(\psi_1) + g(\psi_1) + \epsilon\, q^*(\psi_1, \epsilon) = 0,$$

for each ϵ sufficiently small and letting $\varphi_1 = \psi_1 + \varphi_o$ we have

$$L(\varphi_1 - \varphi_2) + g(\varphi_1 - \varphi_o) + \epsilon\, q(\varphi_1, \epsilon) = 0. \tag{25}$$

But clearly (22) and (23) imply that

$$T_\epsilon(\varphi_1) - \varphi_1 = L(\varphi_1 - \varphi_o) + g(\varphi_1 - \varphi_o) + \epsilon\, q(\varphi_1, \epsilon), \tag{26}$$

so (25) implies $T_\epsilon(\varphi_1) = \varphi_1$. Therefore, since η was chosen arbitrarily and φ_1 is contained in $N_\eta(\varphi_o)$ it follows that every neighborhood of φ_o contains a fixed point of T_ϵ for each ϵ sufficiently small. Since we have that solutions of equation (5) depend continuously on their initial data we may conclude, of course, that every neighborhood of x_o contains a periodic solution for equation (5) of period ω_ϵ for each ϵ sufficiently small and our proof is complete.

Let us now extend somewhat the results of Lemma 3.

Lemma 4. Let the hypotheses of Lemma 3 be satisfied and for each ϵ in $[0, \epsilon_1]$ let θ_ϵ denote the set of points x in $\bar{S}$ such that $f(x, \epsilon) = 0$ and let $\theta = \cup\{\theta_\epsilon : \epsilon \text{ in } [0, \epsilon_1]\}$. Then 0 is contained in θ and the component θ^* of θ containing 0 intersects θ_ϵ for each ϵ sufficiently small. Furthermore, if θ_ϵ is totally disconnected in an open neighborhood of 0 in X for ϵ sufficiently small, then there is a continuous function $\xi(\epsilon)$ mapping some interval $[0, \epsilon_o]$, $\epsilon_o > 0$, into X such that for every ϵ in $[0, \epsilon_o]$, $f(\xi(\epsilon), \epsilon) = 0$.

Proof. It is obvious that $f(0, 0) = 0$ so 0 is contained in θ. Suppose one can select ϵ arbitrarily small so that $\theta^* \cap \theta_\epsilon$ is empty. Then since θ^* and θ_ϵ are closed, there exist $\eta > 0$ such that $N_\eta(\theta^*) \cap N_\eta(\theta_\epsilon)$ is empty. But $N_\eta(0)$ is contained in $N_\eta(\theta^*)$ and there is a neighborhood $N_\mu(0) \subset T_\epsilon(N_\eta(0))$ for all ϵ sufficiently small which, of course, implies $N_\eta(0)$ and θ_ϵ intersect for all ϵ sufficiently small. Thus we have contradicted our suppositions and can therefore conclude the existence of a constant ϵ_o such that for all ϵ in $[0, \epsilon_o]$, $\theta^* \cap \theta_\epsilon$ is not empty. If, in addition, θ_ϵ is totally disconnected in an open neighborhood of 0 in X for ϵ sufficiently small, we may choose ϵ_o such that $\theta^* \cap \theta_\epsilon$ consist of a single element for each ϵ in $[0, \epsilon_o]$. Defining ξ on $[0, \epsilon_o]$ by the formula

$$\xi(\epsilon) = \theta^* \cap \theta_\epsilon,$$

it is clear that ξ is continuous and such that $f(\xi(\epsilon), \epsilon) = 0$. Therefore the proof of our lemma is complete.

Now with Lemma 4 as a tool we extend the results of Theorem 7.

Theorem 8. Let the hypotheses of Theorem 7 be satisfied and let Ψ_ϵ for each ϵ in $[0, \epsilon_1]$ denote the set of periodic solutions for equation (5) of period ω_ϵ, and let

$$\Psi = \bigcup\{\Psi_\epsilon : \epsilon \text{ in } [0, \epsilon_1]\}.$$

There exists a connected subset Ψ^* of Ψ which contains x_0 and intersects Ψ_ϵ for each ϵ sufficiently small. Furthermore, if Ψ_ϵ intersected with some open neighborhood of x_0 is totally disconnected for each ϵ sufficiently small, then there exists a continuous function $\xi(\epsilon)$ mapping some interval $[0, \epsilon_0]$, $\epsilon_0 > 0$, into Ψ such that for every ϵ in $[0, \epsilon_0]$, $\xi(\epsilon)$ is a periodic solution for equation (5) of period ω_ϵ.

Proof. With S defined as in Theorem 7 let θ_ϵ denote the set of points ψ in $\bar{S} - \varphi_0$ such that $\Theta_\epsilon(\psi) = 0$ for each ϵ in $[0, \epsilon_1]$ where Θ_ϵ is as defined by (24). Employing Lemma 4 we have that the component θ^* of the set $\theta = \{\theta_\epsilon : \epsilon$ in $[0, \epsilon_1]\}$ which contains 0 intersects θ_ϵ for each ϵ in some interval $[0, \epsilon_0]$. Hence using (25) and (26) we have $T_\epsilon(\varphi) = \varphi$ for each φ in $\theta^* + \varphi_0$ and ϵ in $[0, \epsilon_0]$. Defining Ψ^* to be the set of solutions of equation (5) corresponding to initial functions in $\theta^* + \varphi_0$ it follows from our hypothesis of continuous dependence on initial data that Ψ^* is connected, contained x_0 and intersects Ψ_ϵ for each ϵ sufficiently small. If Ψ_ϵ intersected with some open neighborhood of x_0 is totally disconnected for each ϵ sufficiently small, then we may choose ϵ_0 so that $\Psi^* \cap \Psi_\epsilon$ consists of a single point for each ϵ in $[0, \epsilon_0]$. Defining the function ξ on $[0, \epsilon_0]$ by the formula

$$\xi(\epsilon) = \Psi^* \cap \Psi_\epsilon,$$

we may observe that ξ is continuous and obviously for each ϵ in $[0, \epsilon_0]$, $\xi(\epsilon)$ is a periodic solution for equation (5) of period ω_ϵ. Therefore our theorem is proved.

We remark that Theorem 7 and Theorem 8 are merely suggestive of the types of results which are possible to obtain concerning the continuity of periodic solutions under perturbations. The very powerful results of Schauder and Leray on the local degree for continuous displacements and the many extensions of this work of more recent origin provide excellent tools for the analysis of such questions. In particular, the theory surrounding the notion of a fixed point index is very useful. The reader interested in such questions of continuity and "nearness" may profitably refer to references [10], [11], [12], [13], [14], [15] and [16].

In conclusion let us mention another interesting and important question associated with periodic behavior for solutions of functional-differential equations of the type considered in this paper. This is, the establishment of the existence of periodic solutions which are necessarily nontrivial in the sense of not being constant functions. We remark that if one imposes a type of uniform instability in a neighborhood of initial functions which yield constant solutions together with the hypotheses of Theorem 5, then it is possible in many situations to establish the existence of nontrivial solutions using the techniques employed in Theorem 2 of [6].

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